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BOUNDARY ELEMENT LATERAL ANALYSIS OF TALL BUILDINGS INCLUDING SOIL-STRUCTURE INTERACTION EFFECTS

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ملخص البحث:
في هذا البحث تم اقتراح طريقة عملية جديدة كفء لأخذ التأثير المتبادل بين التربة والمنشأت المعرضة لأحمال جانبية. تعتمد هذه الطريقة في الاساس علي طريقة التقسم والتي فيها يتم تقسيم المسألة الي جزئين منفصلين. جزء خاص بالجزء العلوي للمنشأ (غير شامل اللبشة) والجزء الاخر جزء خاص بالتربة بالاضافة الي اللبشة. تم استخدام طريقة التكثيف الاستاتيكي علي المستوي المار بين اللبشة والحوائط.وتم ايجاد التشكلات عند هذا المستوي. حاليا يتم استخدام طريقة مشابهة للتكثيف الاستاتيكي ولكن عند المستوي المار بين اللبشة والتربة وهو بدوره يستهلك جهدا ووقتا مقارنة بالطريقة المقترحة.
في هذا البحث تم استخدام الطريقة في التحليل الخطي فقط ومع ذلك يمكن استخدامها في التحليل اللاخطي لاحقا بسهولة وبطريقة مباشرة

Abstract:
A new technique for the analysis of buildings including soil-structure interaction is suggested. The new analysis is based on sub-structuring approach where the system is partitioned into two main parts which are the superstructure part and the raft-soil part. A static condensation technique is implemented at the wall-raft interface. Deformations of supports at wall-raft interface are obtained. Current practical analysis of SSI is implementing the static condensation at the raft-soil interface which is time consuming and tediously job. The new analysis has shown less time and effort in the modeling and analyses. This technique of analysis is presented here only for linear analysis and shear wall buildings. However, this technique can be extended to include nonlinear analysis such as no tension SSI, soil nonlinearity, SSI and analysis of frame buildings.

Introduction
According to many reports [1], construction industry has witnessed a very rapid growth particularly in tall buildings because of tendency to urbanization and industrial development. Considering tall buildings from 5 to 10 stories with raft foundations , it is common – in structural engineering – mainly in the analysis of building at design firms not taking into account the flexibility of the sub structure (foundations and underneath soil) in the analysis of the superstructure. They often carry out design as two parts independently. Generally, buildings are assumed to be fixed or hinged at the ground level. As a consequence, the evaluated responses due to different types of load cases especially the lateral loads (earthquakes and wind loads) do not truly present the accurate behavior of the structure. Raft and soil stiffness will add more flexibility to the structure; so that the overall stiffness will be decreased and a more realistic and economic designs could be achieved. On the other hand, the lateral deflection and the inter-story drift will increase with increasing the soil flexibility.

Although this is more conservative for the structures, the safety of the structure due to lateral deflection should be re-evaluated So, it is important to consider soil-structure interaction in the analysis.

The boundary element method (BEM) [15]:

The boundary element method is a numerical method for solving linear partial differential equations which have been formulated as boundary integral equations. It can be applied in many disciplines in engineering and science including solid mechanics, fluid mechanics, acoustics, electromagnetism, fracture mechanics, and plasticity.

In contrast with other energy methods like finite element method, the boundary element method discretization is only on the problem boundaries. The direct and indirect boundary element methods are the two branches of the BEM. The direct BEM formulates the problem in terms of variables that have definite physical meanings, such as displacements of the boundary nodes of the plate. In contrast, the indirect BEM uses variables whose physical meanings cannot always be clearly specified.

The advantages of the boundary element method are reducing problem dimensionality, requiring low memory, it focuses on the body boundary, good for incompressible materials, easy to define and vary boundary elements, accurate and good for stress concentrations.

In practicality, there are already several programs which are developed using boundary element method and adopted for the solution of many engineering problems [16-19].

The Variational boundary element method (VBEM):

In the direct and indirect boundary element formulations, the variational method and the Galerkin weighted residual approach have been implemented to solve boundary integral equations (BIE). Numerical approaches with a variational basis for the solution of the BIE have been proposed by McDonald [20] and Jeng and Wexler [21, 22]. The modified Galerkin boundary formulations have appeared in literature as well as the Galerkin weighted residual method; which is identical to the discretized system of equations in [20] and in [22]. An algebraic system of equations was revealed by Hsiao, et al. [23] by proposing the Galerkin collocation method. This method proved an asymptotic convergence of the approximate solutions. In references [24, 25,26 and 27] this Galerkin collocation method is clarified with the related error analysis and convergence proofs for different types of problems. Polizzotto [28] and Maier and Polizzotto [29] form a symmetric boundary element formulation by applying the Galerkin method in elasticity. Convergence and stability are proven by Onishi [30] in heat conduction by applying the Galerkin method.

It is relatively recent to employ the variational principles to generate the boundary integral equations. The Hellinger-Reissner and the Wu-Washizu principles were implemented in the late 80's using the variational BEM to develop boundary element formulations. Dumont [31, 32] formulated a hybrid stress boundary element on the Hellinger-Reissner principle. Stresses in the domain and the displacements on the boundary are considered as independent functions in this formulation giving a symmetrical coefficient matrix. The domain variable is approximated in this formulation by using the point force fundamental solution. The solution of potential problems is presented by Defigueiredo and Brebbie [33] by developing an assumed-displacement hybrid boundary formulation based on the generalized variational principle. The generalized variational principle includes three independent variables; two are defined on the boundary and one is in the problem domain. This approach uses the same approximation for the domain variable applied in [31, 32]. This hybrid boundary formulation was later extended to elastostatic problems Defigueiredo and Brebbie [34]. Polizzotto [35] used either the Hu-Washizu principle or the Hellinger- Reissner principle to derive a boundary element formulation which leads to the same symmetric coefficient matrix found by the Galerkin approach presented in [29]. He also proposed the boundary min-max principle, which is a boundary variational principle, where the boundary displacements and tractions are considered as the independent field variables. The resulting boundary element formulation is shown to be equivalent to the previous Galerkin approach, which shows symmetry and definiteness. This formulation has also been extended to include elastoplasticity [35] and elastodynamics [36]. Recently, Felippa [37] proposed a formulation for boundary elements in linear eleatostarics, which is based on his parametrized displacement generalized hybrid variational principle by McDonald et al. [38]. Distinct boundary element models can then be obtained by setting different values to a parameter. Therefore his approach can be considered a systematic way of deriving boundary element formulations.

The analysis using the variational boundary element based shell elements is derived based on minimizing the relevant energy function by Naga [39]. The new variational BEM provides solutions that approach the analytical solution without the requirement of vast descirtization. The results from this method approach the analytical solution from less discitization than both the FEM and the present BEM. The main disadvantage of this method is that the mathematical derivation of the kernals is much more tedious. This method provides advantages from both the BEM and the FEM. These advantages include the descritization of the domain as well as producing stiffness matrices which are symmetric. Naga [39] developed the variational BEM formulation for 2-D elasticity and plate bending which has been coupled to solve shell elements. The variational BEM has been programmed and tested [39].

Sources of Soil structure interaction:

In the structural analysis, the assumption of fixed base for the building especially for the building on soft or medium soil is not realistic [2-14]. Usually designers are assuming fixed or hinged base for the sake of simplicity. This assumption may be accepted if the structure will be constructed on rock layer or if the relative stiffness of the substructure (soil-foundation system) compared to the superstructure is high.

In the most occasions, existence of soil induces two separate effects on the structure, first, the amplitude and the direction of the free ground motion from the bed rock will be adjusted (amplified or degraded) at the level of structure’s base [3]. Second, hence the underneath soil is flexible, the forces acting on the masses of the structures will enforce the supporting systems (raft foundation..etc.) and the underneath soil to deform. These two phenomena are referred to as kinematic interaction and inertial interaction, respectively. In fact, kinematic interaction is the inability of the foundation to match the soil deformations due to the free field ground motion. On the other hand, the inertial interaction can change the structure periodic time (T) hence the structure response to the seismic loading.

The Proposed New Technique

In this section, the proposed technique will be discussed in details. The analysis will pass with 3 main stages:

1- Pre processing
   a) Super Structure: after drawing the building in ETABS, a nine text files will be prepared with the help of ETABS program to calculate the stiffness.
   b) Sub structure: the raft with walls locations will be drawn in PLPAK to get the stiffness.

2- Analysis :
   a) Super Structure: VBEM program will be run to get the stiffness of the super structure (K super). Then the stiffness will be condensed at supports level.
   b) Sub structure: PLPAK program including stiffness manager program will be run to calculate the condensed stiffness of the sub structure (K sub).
   c) K total condensed = Ksuper + Ksub
   d) Displacements of supports U = K-1total * F (where F= reactions from super structure).

3- Post Processing:
   a) Displacements of the all joints of structure easily can be obtained.
   b) Internal forces at any joint also can be obtained.

The developed technique is illustrated in detail followed by an illustrative example. The entire methodology of the work is illustrated in details with flow chart presentation.

Analysis of Super Structure with the VBEM program

Most civil engineering structures are subdivided into plate bending, 2-D elasticity and beam element problems. Plate bending problems have three degrees of freedom per node; one displacement in the plane perpendicular to the plate and two rotational degrees in the plane of the element. 2-D elasticity problems have two degrees of freedom per node; two in displacements parallel to the element at each node. Three dimensional beam elements have six degrees of freedom per node; three rotational and three displacements. The problems are analyzed and the solution is coupled [40].

There is a program called (Shell Element Program) which solves both 2-D elasticity and plate bending using the developed variational boundary element method formulation [39].

The shell element consists of the superposition of the plate bending program (2-D elasticity) and the membrane element program (plate bending).

Shell Element Program used for analysis of super structure with fixed supports

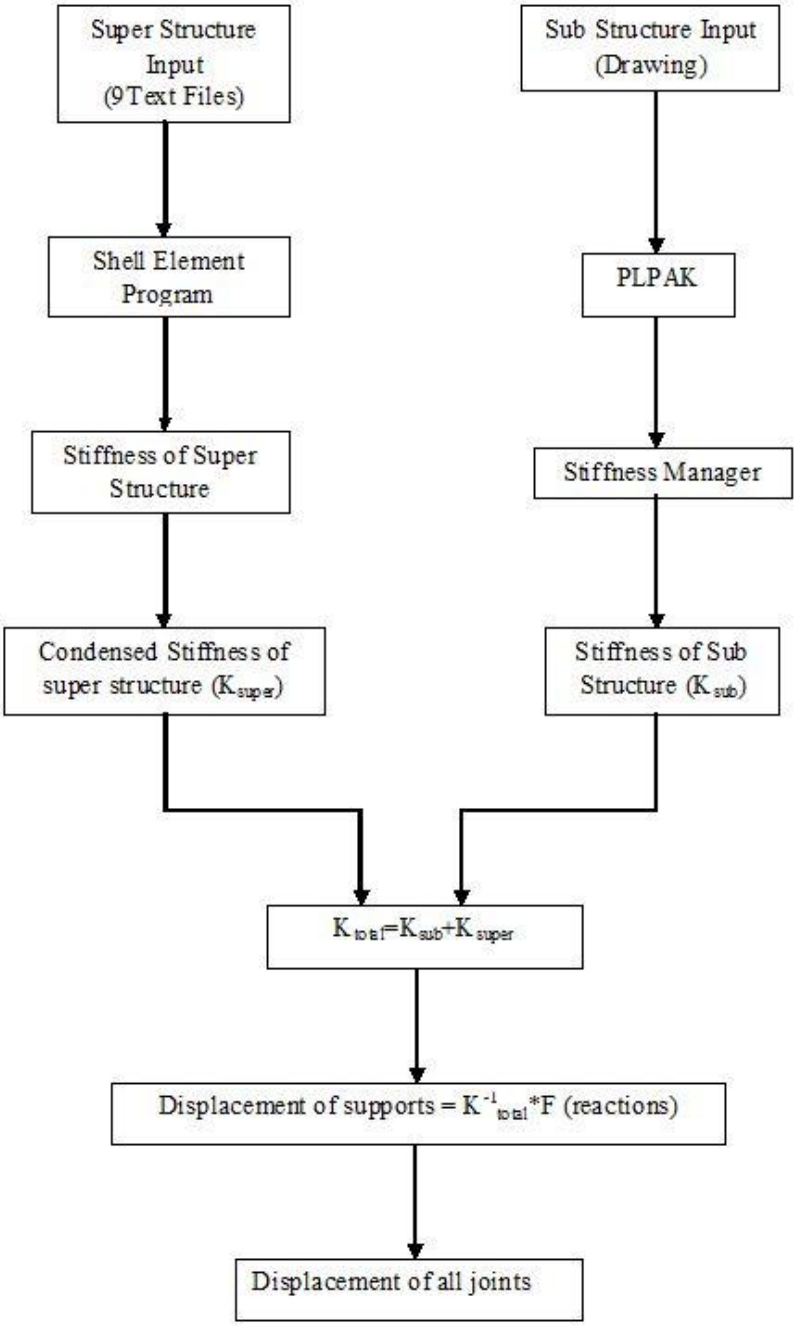

Figure 1 Flow chart illustrating the sequence of solution.

Shell Element Program:

Input Files

There are 9 input files for the Shell Element Program as follows:

1- GFILE.txt - General File
   This includes all data of the problem
2- PCOOR.txt Point Coordinates
   This includes the coordinates of all the nodes
3- PASSL.txt - Point Assigned Loads
4- PASSD.txt - Point Restraints
5- ARASE.txt - Area Section

6- ARASL.txt - Area Loads
7- ARCO.txt - Area Connectivity
8- NDIV.txt - Division of the plates
9- PINT.txt - Internal Points

Output Files

1- OUTFILE

A file containing all problem data (point's coordinates, shells loads and definitions…), support reactions and points displacements.

2- KMATRIX&LOAD VECTOR.xls

A file containing the stiffness matrix of the super structure.

And there are another file but not needed for lateral analysis of shear wall buildings.

Analysis of Sub Structure with the PLPAK program

PLGEN model is generated with raft with walls locations and without wall reactions, then BE-Files will be extracted from PLGEN model and using the Stiffness Manager, the stiffness of the sub structure will be computed and the nodes of the walls.

Illustrative Example

In this section, an illustrative example is presented with detailed steps. This is to show the process to get the stiffness of the super-structure by shell element program and construct the raft-soil PLGEN model in PLPAK.

1- Structural drawings: drawing the required building in CAD format to determine dimensions and position of all structural elements. It is the key to begin the modeling on SAP as shown in figure 3.8.
2- SAP 3D Model: we will make a 3D model with the building as shown in figure 3.9 to make it easy for extracting the 9 input files to shell element program and to compare results later. SAP model has to be modeled in a correct way. All loads should be included. And the end conditions for each wall will be fixed.
3- Extracting data which help in creating the 9 text files (from display menu then show tables) as shown in figure 3.10
4- PLPAK model: In this step, PLGEN model is generated with raft with walls locations and without wall reactions. The model is shown in figure 3.12-3.13.

Methodology

In this section, a flow chart describes the entire process of including soil-structure interaction in the analysis of the building is presented as shown in figure 3.14. A detailed description of the function of each part is presented. The proposed new technique can be summarized in the following steps. These steps can generally be divided into 4 main groups of steps.

a. The preparation of the models

This includes the preparation of the SAP and PLPAK model and shell element program.

1- Prepare the structural drawing as discussed in section 3.3-1.
2- Create SAP model as discussed in section 3.3-2.
3- Export the required tables listed below:
a. Joint Coordinates
b. Connectivity – Area
c. Joint Restraint Assignments

d. Joint Loads
e. Area Section Assignments
f. Area Loads Assignments

4- The 9 input files to shell element program will be prepared as follows and as discussed in section 3.2.1.1:

- Joint Coordinates Table will be copied into PCOOR.txt file
- Connectivity – Area Table will be copied into ARCO.txt file
- Joint Restraint Assignments Table will be modified as follows, Yes will convert to 0 then copied into PASSD.txt file
- Joint Loads Table will be copied into PASSL.txt file
- Area Section Assignments Table will be copied into ARASE.txt file
- Area Loads Assignments Table will be copied into ARASL.txt file

- According to the other 3 files (GFILE.txt, NDIV.txt, and PINT.txt) there is no need to use SAP in filling them and can be filled easily.

5- Create the raft model on PLGEN. This model includes only the raft, soil data and piles if found. No loads are included in this step as described in section 2.8.6.

b. Analysis

1- Run shell element program with the 9 text files then the OUTFILE and KMATRIX&LOAD VECTOR.xls will be created.
2- Stiffness matrix of the super structure (K) will be obtained from KMATRIX&LOAD VECTOR.xls file.
3- Deformations of joints (U) (3 dicplacements Ux,Uy,Uz and 3 rotations Rx, Ry, Rz) will be obtained from OUTFILE (note that deformations at all supports equal to zero).
4- Load Vector can be obtained from P= K * U
5- Stiffness Matrix of the super structure (K) will be condensed at supports. Condensed Stiffness Matrix of the super structure (Ksuper).
6- Extract BE-Files form PLGEN model.
7- Put the path of the file with the name (Ex.1LoadCase1.in) into the file with name (PATHIN.txt) then run stiffness manager.exe
8- The condensed stiffness matrix of sub structure (Ksub) now is ready in the file with name (Ex.1LoadCase1.in.$S3-K$).
9- Condensed stiffness matrix of the structure Ktotal = Ksuper + Ksub
10- By knowing condensed stiffness matrix of the structure (Ktotal) and Load vector at supports (Pc which is a part of P obtained from step 4) we can get deformations at supports (Uc) = K-1total * -Pc

c. Post processing

1- Deformations of other joints can be obtained as follows:

$$\left[\begin{array}{c|c} K_{cc} & K_{ce} \\ \hline K_{ec} & K_{ee} \end{array}\right] \times \left[\begin{array}{c} U_c \\ \hline U_e \end{array}\right] = \left[\begin{array}{c} P_c \\ \hline P_e \end{array}\right]$$

K (Stiffness matrix of the super structure) x U(Deformations of joints) = P(Load Vector). Kcc,Kce,Kec and Kee are the components of the stiffness matrix obtained from step 3.4.b.3, Uc is obtained from step 3.4.b.10, Pc and Pe are the components of the load vector obtained in step 3.4.b.4 . The only unknown term is Ue (joints deformations of the super structure) Ue = Kee-1 x (Pe – Kec x Uc)

Data

It is a 9 Story shear wall building subjected to static lateral force and vertical load. In this example, a typical shear wall building shown in figure 4.11 consists of 9 stories, 4 meter for the first story and 3m for the 8 remaining stories. This structure is subjected to static lateral loading. A 10 KN/m static lateral distributed load is applied in the positive direction of X- axis and 5 KN/m2 distributed vertical load in gravity direction. All shear walls of the building are 0.15 x 2 m. Also, all slabs are 0.2 m as shown in figure 4.1. The raft foundation thickness is 0.9 m. Analyses were carried out using substructuring approach where the super structure is modeled using SAP - the well-known finite element commercial program – figure 4.2 then analyzed using shell element program. The raft foundation and the underneath soil are modeled using PLPAK – a developed package for solving shear deformable plates on Winkler and elastic half space based on boundary element method and stiffness method figures 4.3 and 4.4. Lateral deflection, lateral deflection ratios (SSI/NSSI), Inter story drift and drift ratios (NSSI/SSI) are obtained. The results are compared against the same structure modeled using direct method. SAP2000 – finite element well known commercial program is used figures 4.5 and 4.6 – where the piles modeled as a springs with a vertical stiffness, horizontal rotational stiffness in X and Y equal to (1014473KN/m, 63404 KN.m/rad and 63404 KN.m/rad) respectively, assuming that there are no displacement in X and Y directions and no torsion will occur to the substructure.

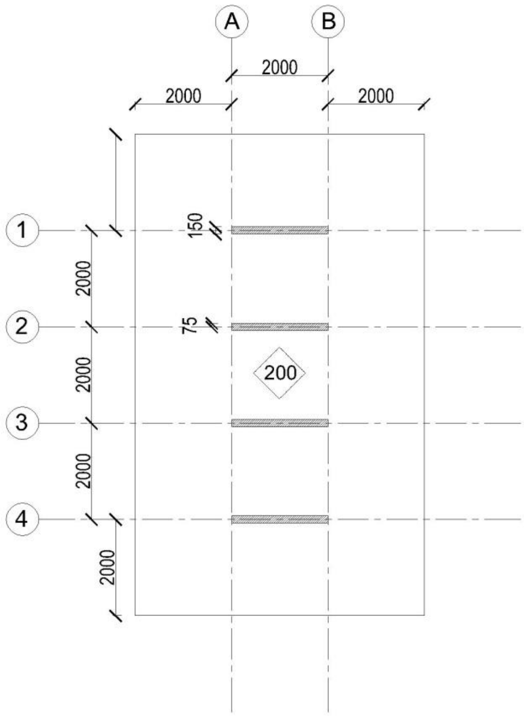

Figure 2 Example plan

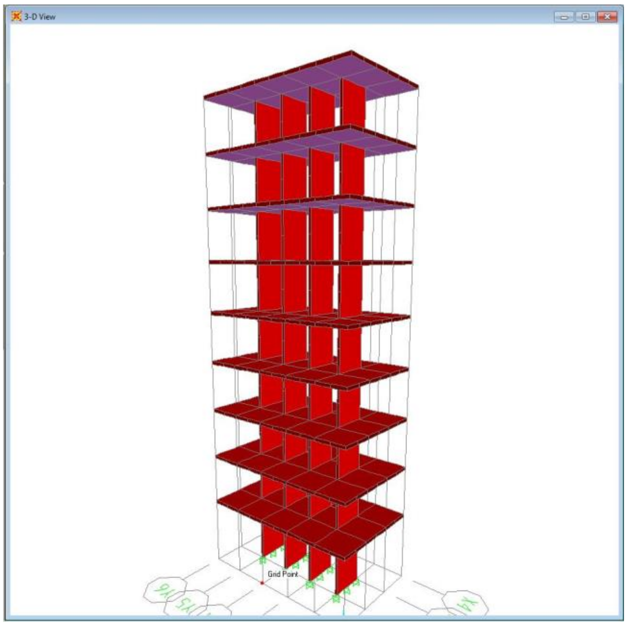

Figure 3 The SAP2000 3D model of super structure.

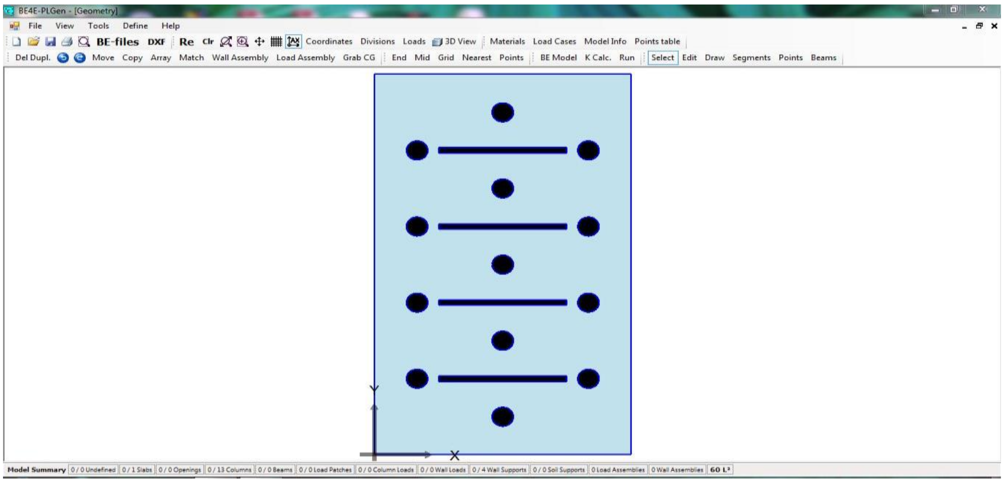

Figure 4 The raft model in PLGEN.

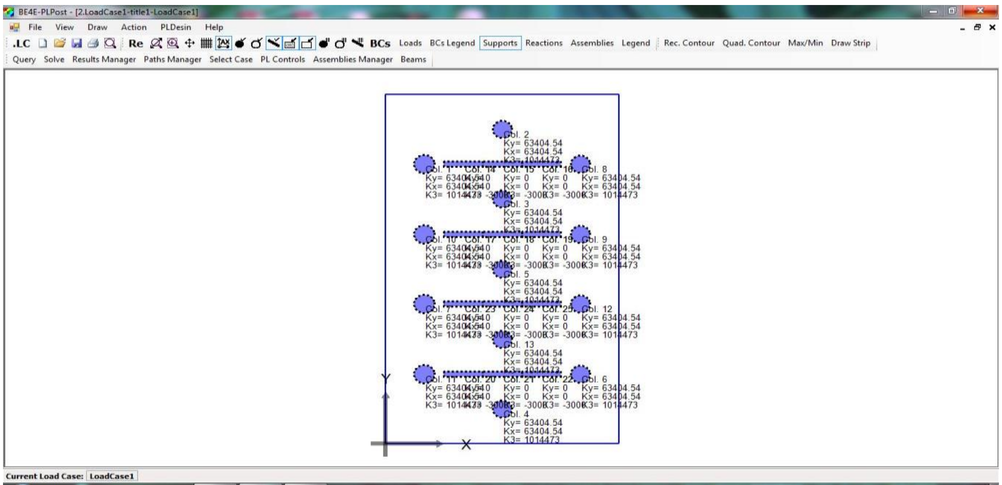

Figure 5 The raft model in PLPost.

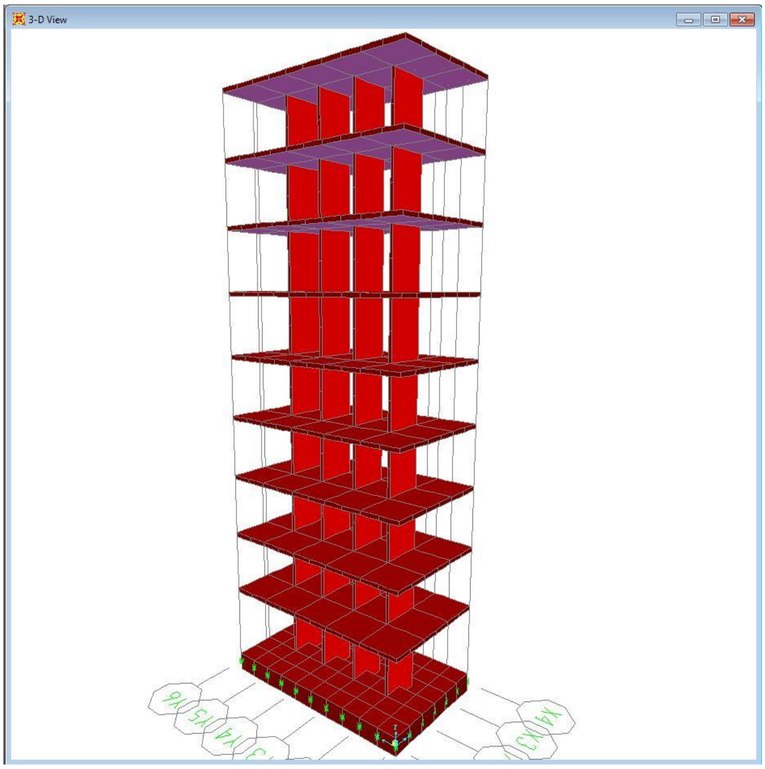

Figure 6 The SAP2000 3D model of the whole structure including soil – Direct Method.

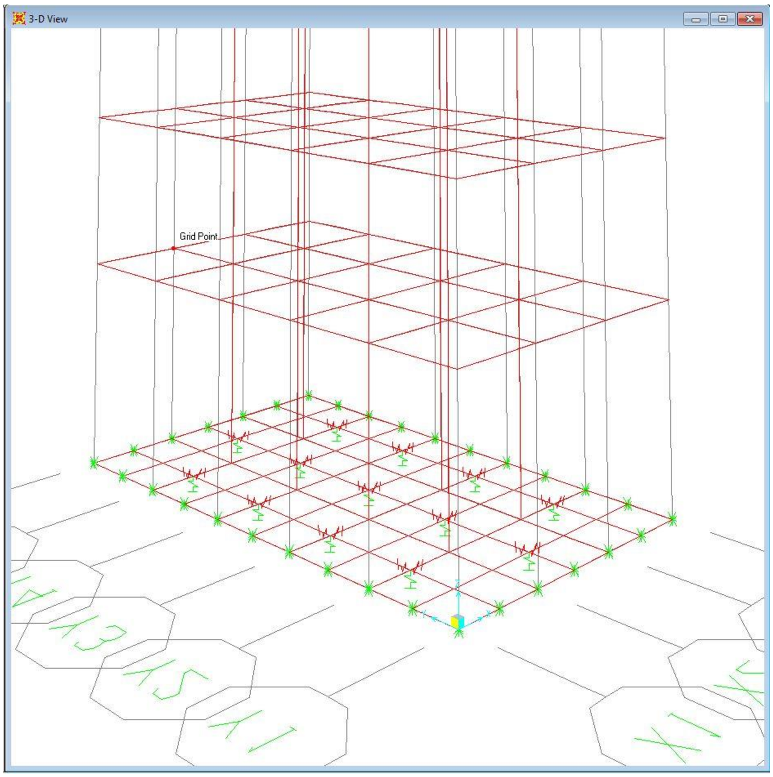

Figure 7 The SAP2000 3D model of the whole structure including soil – Direct Method.

I. Results

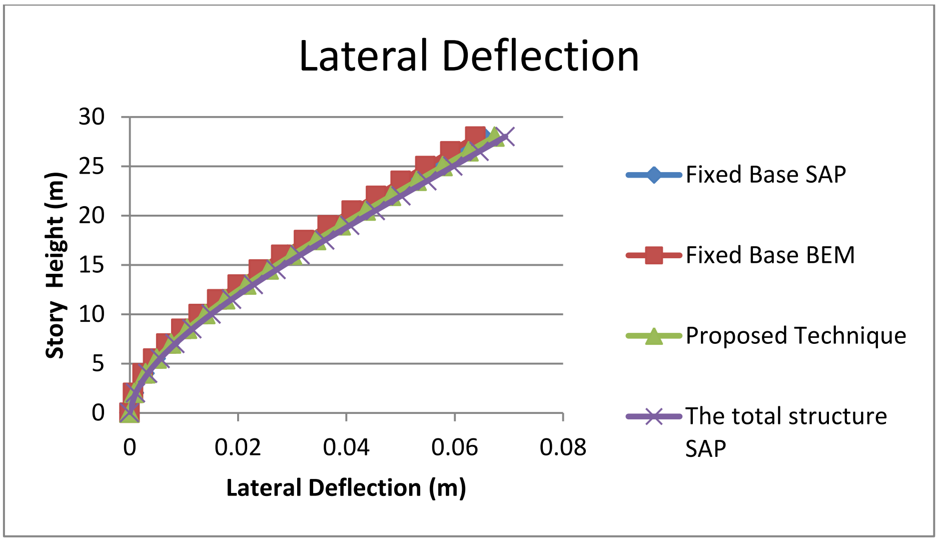

Figure 8 Lateral Deflection in X-direction

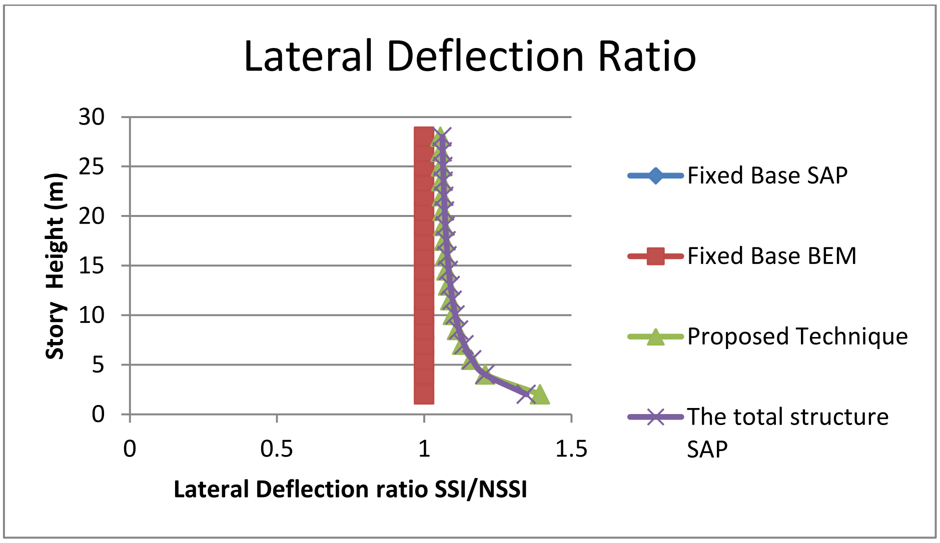

Figure 9 Lateral Deflection Ratio SSI/NSSI in X-direction

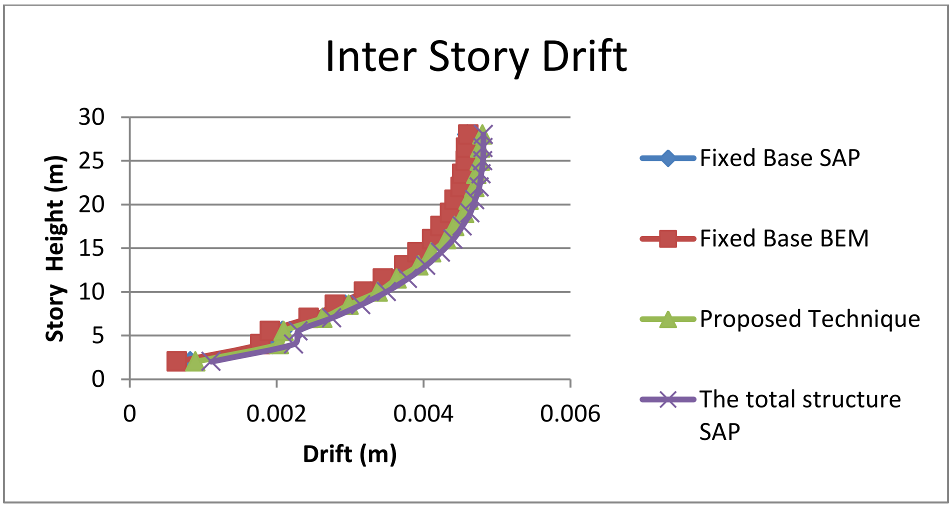

Figure 10 Inter Story Drift in X-direction

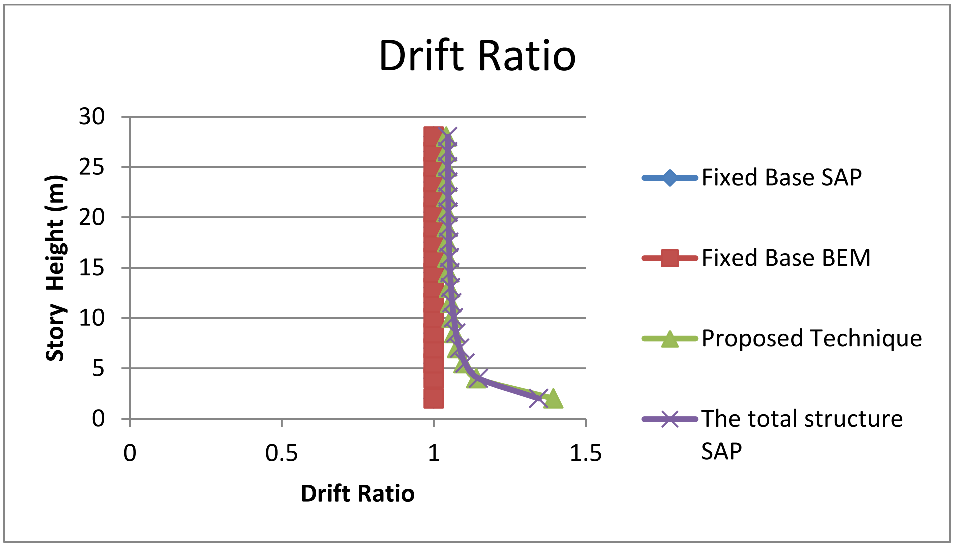

Figure 11 Drift Ratio SSI/NSSI in X-direction

II. Conclusions

The behaviors of tall building with fixed support and without fixed but resting on the piled raft with consideration of soil interaction have been studied in the present paper. In this we get the results from analysis of model for fixed base (SAP and BEM) they show the less lateral deflection, It is concluded that the soil interaction effect is very remarkable. Lateral deflection of the building increases when analyzing the tall building with SSI method. Finally, it can be concluded that although conventional design procedure omitting SSI is conservative it is required to ensure the structural safety due to lateral deflection.

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